Handout for Section 3.1

Length of an Arc

Consider the following diagram:

![Diagram of a circle with an arc and angle θ]

If the measure of angle θ is given in radians, then, by the definition of radian, we get \( \theta = \frac{s}{r} \).

Notice that if the arc length \( s = 2\pi r \) (the entire circumference), then we obtain

\[
\frac{s}{r} = 2\pi \quad \text{(by dividing through by } r) \]

\[
\theta = 2\pi \quad \text{(by substitution)}
\]

This shows that a central angle \( \theta \) that corresponds to 1 revolution has measure \( 2\pi \) radians.

The equation \( \theta = \frac{s}{r} \) can be solved in terms of s:

\[
s = r\theta,
\]

and this equation can be used to solve problems of the following type.

**Example 2** A wheel of radius 12 cm. Rolls through \( 1\frac{1}{2} \) revolutions. How far does the wheel travel?

**Solution**

The problem states that the wheel rolls \( 1\frac{1}{2} \) revolutions. This implies an angle. \( 1\frac{1}{2} \) revolutions can be written as \( \frac{3}{2} \) revolutions. Every revolution determines an angle with measure \( 2\pi \) radians.

Thus, we get the ratio \( \frac{2\pi}{1 \text{ revolution}} = 1 \). The \( \frac{3}{2} \) revolutions can then be converted to radians as follows:

\[
\left( \frac{3 \text{ revolutions}}{2} \right) \left( \frac{2\pi}{1 \text{ revolution}} \right) = 3\pi
\]
Therefore, \( \theta = 3\pi \)

Substituting the values for \( r \) and \( \theta \) into the equation \( s = r\theta \), yields the desired distance:

\[
\begin{align*}
  s &= r\theta \\
  s &= (12\text{cm.})(3\pi) \\
  s &= 36\pi \text{ cm.} \\
  s &\approx 113.1\text{ cm.}
\end{align*}
\]

**Example 2** The minute hand of a clock is 4.8 inches long. How far does its tip move in 45 minutes? Give your answer correct to one decimal place.

![Clock with minute hand](image)

**Solution**

Since the minute hand of any clock completes one revolution in 60 minutes, then it completes \( \frac{45}{60} = \frac{3}{4} \) of a revolution in 45 minutes.

In order to be able to use the formula \( s = r\theta \), the measure of angle \( \theta \) must be in radians, so we convert as follows:

\[
\left( \frac{3 \text{ rev}}{4} \right) \left( \frac{2\pi \text{ rad}}{1 \text{ rev}} \right) = \frac{3\pi}{2}
\]

[Notice that it is customary not to write the word radian. When the units are omitted in an angle measure, the assumption is that the measure is given in radians]

\( \theta \) must be in radians; \( r = 4.8 \)

Since 60 is one revolution of the minute hand, \( \frac{45\text{ min}}{60\text{ min}} \) is the fractional part of one revolution. Since one revolution = \( 2\pi \), then \( \theta = \frac{45}{60} \cdot 1 \text{ rev} \) or \( \theta = \frac{45}{60} \cdot 2\pi \).
\[ \theta = \frac{45}{60} \cdot 2\pi \]
\[ \theta = 4.7 \]

\[ s = r\theta \]
\[ s = (4.8)(4.7) \]
\[ s = 22.6 \text{ inches} \]

**Example 2**  A space shuttle 365 miles above the earth orbits every 7 hours. How far, to the nearest 10 miles, does the shuttle travel in 3 hours? (The diameter of the earth is approximately 8000 miles).

\[ r = 4365 \]
\[ \theta = \frac{3}{7} \cdot 2\pi \]
\[ \theta = 2.7 \]
\[ s = r\theta \]
\[ s = (4365)(2.7) \]
\[ s = 11750 \text{ miles} \]

**Example 3**  A pendulum is 11 cm long and swings through an arc of 193° in 6 seconds. Find the Distance the pendulum travels in 3 seconds.
\[ r = 11 \]

\[ \theta = \frac{193^\circ \cdot \pi}{180^\circ} \]
\[ \theta = 3.4 \]

\[ s = r\theta \]
\[ s = (11)(3.4) \]
\[ s = 37.4 \text{ cm} \]

In 3 seconds; \[ s = \frac{37.4}{2} = 18.7 \text{ cm} \].

Because \[ 2\pi = 1 \text{ rev} \], \[ \frac{2\pi}{1 \text{ rev}} = 1 \].

**Example 4** A certain automobile manufacturer’s testing indicates that a tire will wear out in \(3.60 \times 10^7\) revolutions. Assuming the diameter of the tire is 14 inches, how many miles can the owner drive the car before he changes a tire? (Round off answers to the nearest mile.)

**Solution:** The total distance traveled is actually the arc length that the tire “traces on the road”. We then need to look for two things: the angle and the radius.

First, the number of revolutions will give us the angle in radians.

\[ \theta = 3.60 \times 10^7 \text{ rev} \]
\[ \theta = 3.60 \times 10^7 \text{ rev} \left( \frac{2\pi}{1 \text{ rev}} \right) \]
\[ \theta = 2.262952 \times 10^6 \]

How can you multiply just one side of the equation by \( \frac{2\pi}{1 \text{ rev}} \)?

\( \frac{2\pi}{1 \text{ rev}} \) is a conversion factor.

The radius of the tire is:

\[ r = 7 \text{ in} \]

From which we find:
\[ s = r \theta \]
\[ s = (2.261952 \times 10^8)(7) \left( \frac{1\, ft}{12\, in} \right) \left( \frac{1\, mi}{5280\, ft} \right) \]
\[ s = 24990 \text{ miles} \]

**Problems:**

1. How far does the tip of a minute hand measuring 2 cm move in 3 hours?

2. A 50 cm pendulum on a clock swings through an angle of 100°. How far does the tip travel in one arc?

3. If a penny rolls downhill for 300 meters and has a diameter of 1.5 cm, how many revolutions did the penny make?

4. A winch is connected to a length of rope measuring 15 ft. If the radius of the winch is 5 in, how many revolutions did the winch make to pull in all of the rope?

5. The minute hand of a clock is 4.8 inches long. How far does the tip travel in 40 minutes?

6. A winch with a 3-inch diameter is connected to a rope of length 10 feet. If the winch makes one revolution every 10 seconds, how long will it take to roll up the rope?

**Answers**

1) 37.68 cm  2) 87.3 cm  3) 6369.4 rev  
4) 5.73 rev  5) 20.11 in  6) 127.3

**Linear and Angular Velocity**

In Intermediate Algebra, we learned that velocity is the distance traveled (s) divided by the time elapsed (t). If you ride your bike at a constant speed, after 4 hours you travel 20 miles, then your velocity is 5 miles per hour.

\[
\text{Linear velocity} = \frac{s}{t}
\]

\[
\text{Linear velocity} = \frac{20\text{miles}}{4\text{hours}} = \frac{5\text{miles}}{\text{hour}}
\]

We define velocity with two elements: direction and magnitude. That is, we have to know where an object is going and “how fast” it is getting there. The velocity of an eastbound truck traveling at 55 mph on a long STRAIGHT freeway is, as you have guessed, 55 mph east!

Linear velocity measures the distance covered per unit per time. But not all things go in straight paths. Sometimes, we get what is known as uniform circular motion. In circular motion, the distance is the arc length.

For an object traveling in a circular path, the velocity is not constant. This is so because even if the “fastness” is the same, the direction is constantly changing. (Incidentally, we call the “fastness” component of the velocity, speed.)
Just as we define linear velocity as distance traveled divided by the time, we define angular velocity as the amount of rotation per unit of time.

\[
\text{Angular velocity} = \omega = \frac{\theta}{t}
\]

Where: \( \theta \) is the angle of rotation in radians and \( t \) is the time elapsed.

**Example 1**

Let’s say you are a runner on a circular track like this:

![Figure 1](image.jpg)

If you had to run the entire length of the track (a fixed angle) in a fixed amount of time, say 2 minutes, which lane would you choose? (I don’t know about you but I would choose lane A, if I were running. This lane would allow me to casually traverse the entire length of the track while my opponents sweat it out!)

So, the nearer you are to the center of the circle, the slower you need to run. Consequently, you need to run faster if you are farther away from the center of the circle.

Expressed another way: **linear velocity is directly proportional to radius** given a fixed angle and a fixed time.

Say you had to run lane C in a fixed amount of time. As your velocity decreases, so does the length of track you traverse (along with the angle the arc length subtends.) We can see that the greater the angle we have to subtend by the length of the track we traverse, the greater the velocity we need. If we want a smaller angle, we can decrease our velocity.

So, **linear velocity is proportional to the angle** given a fixed radius and a fixed time.

If the contest was based solely on the amount of time a runner takes to traverse say lane C, doesn’t it make sense that if you want to have the least amount of time, you would have to run faster? We would conclude that someone who finishes the course in 1 minute is a faster runner than someone who finishes the course in 20 minutes.

So velocity increases as time decreases. And as time increases, velocity decreases.

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Then, velocity is inversely proportional to time given a fixed angle and a fixed radius.

**Putting it together.** We have seen how linear velocity is proportional to the radius and to the angle subtended by the arc length (or the angle of rotation). And, it is inversely proportional to the time. All of these quantities can be placed in a single equation. First we start with our definition of velocity:

\[
v = \frac{s}{t}
\]

Are the fractions in the parentheses all conversion factors?

\[
v = \frac{r\theta}{t} \quad \text{because } s = r\theta
\]

In this case, they are. They will help us get ft/sec.

\[
v = r\omega \quad \text{because } \omega = \frac{\theta}{t}
\]

Example 2 Isaac Newton wanted to explain the motion of the planets around the sun to his young pal, Albert Einstein. He got a stone from the garden and tied it with a piece of worn-out rope so that he had 40 inches between his hand and the stone. Using his head as the sun, he spun the stone over his head at 120 rpm. Al screamed with delight. Suddenly, the rope broke! At what velocity will the stone hit Al right between the eyes? Now we see why Al wears his hair the way he does. (Give answer correct to two decimal places in ft/sec.)

**Solution:** Because \( v = r\omega \), we need to find both \( r \) and \( \omega \).

First, we’ll find \( \omega \).

\[
\omega = \frac{120 \text{ rev}}{\text{min}}
\]

\[
\omega = \frac{120 \text{ rev}}{\text{min}} \cdot \frac{2\pi}{1 \text{ rev}} \cdot \frac{1 \text{ min}}{60 \text{ sec}}
\]

\[
\omega = \frac{12.5664}{\text{sec}}
\]

Before we find \( v \), we’ll express \( r \) in feet instead of inches.

\[
r = 40\text{ in} \cdot \frac{1 \text{ ft}}{12 \text{ in}} = \frac{10 \text{ ft}}{3}
\]

Now we are ready to solve for the velocity.

\[
v = r\omega
\]

\[
\approx 41.89 \frac{\text{ft}}{\text{sec}}
\]

Note: a conversion factor is equivalent to multiplication by one.

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**Example 3** You are the chief science officer of the constellation-class starship U.S.S. Learner. Your vessel has just established orbit at 500 km over the surface of a newly discovered and, as yet, unnamed class-M planet. You are told that you make 1 revolution every 4 hours. Sensors indicate a linear velocity of 1990 km/hr. Starfleet wants you to state the radius of the planet in your report. (Give the answer to the nearest kilometer.)

**Solution:** The linear velocity of your starship is determined by its angular velocity and the radius of the circle it traces. That “big radius” is the sum of the radius of the planet and your distance from the planet’s surface.

So we let \( x \) = radius of the planet

\[
x = r + 500 \text{km}
\]

“\( r \)” is the ‘big radius’.

Now, we determine the ship’s angular velocity.

\[
\omega = \left( \frac{1 \text{ rev}}{4 \text{ hours}} \right) \left( \frac{2\pi}{1 \text{ rev}} \right) = \frac{\pi}{2 \text{ hours}}
\]

Now we know the ship’s linear velocity.

\[
v = \frac{1990 \text{km}}{\text{hr}}
\]

\[
v = r\omega
\]

\[
r = \frac{v}{\omega}
\]

Dividing both sides by \( \omega \).

Substitute the proper values and solve for \( x \).

\[
x = \left( \frac{1990 \text{km}}{\text{hr}} \right) \frac{\pi}{2 \text{hr}}
\]

\[
x = \frac{3980 \text{km}}{\pi}
\]

\[
x \approx 767 \text{km}
\]

**Example 4** The center field of a circular track has a radius of 20 feet. There are three tracks that go around this center field with A as the innermost lane. Each runner is required to stay at the center of each lane. We have seen (in figure 1) that it is easy for runners to cheat by running on lane A which has the least amount of track to cover. In their desire to be fair to all runners, the judges have decided to give the runners on lanes B and C head starts so that all runners will have to cover the same distance. If all the lanes are 3 feet wide, how much of a head start in radians should the judges give these runners? (Give answers to the nearest tenth of a foot.)

**Solution:** If the tracks are 3 feet wide, then in order to stay at the center of the track, the players would have to stay at an imaginary center line which is 1.5 feet at the edge of the lane. The radius of Lane A, then would have to be 20ft + 1.5ft = 21.5ft. Lane B would have a radius of 20ft + 3ft (the width of lane A) + 1.5ft = 24.5ft. Lane C, 20ft + 6ft (combined widths of lanes A and B) + 1.5ft = 27.5ft.

The shortest path will be in lane A. If the runner on lane A makes a complete circle, then the path he will cover is given by the equation \( s = r\theta \).
\[
\theta = 2\pi \\
s = (21.5 \text{ ft})(2\pi) \\
A = 135.09 \text{ ft}
\]

We want the distance covered by runner B equal to runner A’s distance. Since we know that the radius for lane B is 24.5 ft and the arc length is 13.09 ft, we can solve for the angle using \( \theta = \frac{s}{r} \).

\[
\theta = \frac{135.09 \text{ ft}}{24.5 \text{ ft}} = 5.514 \text{ ft} \\
B = 5.514 \text{ ft}
\]

Doing the same thing for lane C, we get:

\[
\theta = \frac{135.09 \text{ ft}}{27.5 \text{ ft}} = 4.912 \text{ ft} \\
C = 4.912 \text{ ft}
\]

The head starts are given by subtracting the angles from \( 2\pi \).

Lane B: \( 2\pi - 1.76\pi = 0.24\pi \)

Lane C: \( 2\pi - 1.56\pi = 0.44\pi \)

**Problems:** (Unless otherwise specified, answers should be rounded off to the nearest tenth in the units given. And remember, the angles used must be in radians.)

1. If an ant at the tip of a clock’s 5-in second hand were to suddenly lose its grip (on all 6 legs), how fast would it fly off? (Give answer in inches per second.)

2. A motorist enters a freeway on-ramp of radius 150 ft. A sign says the speed limit is 25 mph. Provided the motorist adheres to the speed limit, what is his angular velocity?

3. A clock’s 2-ft long pendulum wings from rest off to one side at an angle of 15°. If it travels to the opposite side in 1 sec., what is the bob’s linear velocity?

4. A roll of tape on a dispenser has a radius of 1.5 inches. If someone pulls out tape at a rate of 2 in/sec, what is the roll’s angular velocity? (Assume the radius of the roll of tape doesn’t change while the tape is being taken from it.)

5. What is the linear velocity of a point at the edge of a 12-inch 33 rpm record?

6. A barbell with a pair of 25 kg weights on either side having a radius of 6 inches is given a kick and rolls 1 yard in 3 secs. What is the angular velocity of a point on the edge of one of the weights?

7. At a horse race, three horses and their riders started out traveling in a single row. As improbable as it may seem, these three horses ran neck and neck throughout the race, and even reached the finish line at the same time. At one point, the horses had to make a turn
along the edge of a semicircular patch of land with a radius of 16 feet. If each horse occupied an imaginary track 6 feet wide, how fast did each horse run in order to have made the turn in 1 minute? (Compare Example 5.)

**Answers**

1) \(0.5\) in/sec
2) \(880\) rads/hour
3) \(1.0\) ft/sec
4) \(1.3\) rads/sec
5) \(1244.1\) in/min
6) \(2\) rads/sec
7) \(50.3\) ft/min, \(69.1\) ft/min, \(88\) ft/min