10.2b Applications

The cross sections of a television antenna dish are in parabolic shape. When a television signal hits the antenna dish, the parabolic shape of the dish reflects the signal inward to the focus of the parabola. The signal is then transferred to the receiving equipment.

Definition:

1. The line segment that passes through the focus of a parabola and has endpoints on the parabola is called a focal chord. The focal chord perpendicular to the axis of a parabola is called the latus rectum.
2. A line is tangent to a parabola at a point on the parabola if the line intersects, but does not cross, the parabola at the point

Reflective Property of a Parabola: The tangent line to a parabola at a point P makes equal angles with the following two axes

a. The line passing through P and the focus
b. The axis of the parabola

Thus, the distance between the tangent point and the focus is the same as the distance from focus to the point where the axis and the tangent line intersect.

Example 1: Find an equation of the tangent line to the parabola given by \( x^2 - 4y = 0 \) at the point \( (4, 4) \)

Solution: The equation is the same as \( (x - 0)^2 = 4p(y - 0) \), by comparing to the standard equation of the parabola, we see \( (x - h)^2 = 4(y - k) \), we get \( h = 0, k = 0 \).

The parabola has the coordinate \((0, 0)\) as vertex. Since \(4p = 4\), we get \( p = 1\). Thus the focus is \((0, 1)\)

The parabola has the y-axis as axis, i.e. \( x = 0 \) is the axis.

From the reflective property of a parabola, we know the distance between the focus \((0, 1)\) and \((4, 4)\) is the same as the distance from focus \((0, 1)\) and the intersection of axis and tangent line. Let \((0, b)\) be the point of the intercept of the axis and tangent line. Therefore, we have

\[
\sqrt{(4-0)^2+(4-1)^2} = 5 \quad \text{(distance between focus} \,(0, 0)\text{ and} \,(4, 4)\text{)}
\]

That means the distance from \((0, 1)\) to \((0, b)\) is 5, therefore \(b = -4\) (count 5 down from \((0, 1)\))

We now find the other point \((0, -4)\) on the tangent line, with two points \((4, 4)\) and \((0, -4)\), we can construct the tangent line as

\[
y - 4 = \frac{4 - (-4)}{4 - 0}(x - 4) = \frac{8}{4}(x - 4) = 2(x - 4)
\]
Thus \( y = 2x - 4 \) is the equation for tangent line

**More examples of finding the standard equations of a parabola**

**Example 2:** Find the standard equation of a parabola with vertex (0, 0) and focus at (0, -2)

**Solution:** The axis of the parabola is vertical line passing through the vertex (0, 0) and focus (0, -2), therefore the axis is vertical, thus the standard form for this parabola is \((x - h)^2 = 4p(y - k)\)

vertex = \((h, k) = (0, 0) \Rightarrow h = 0, y = 0\)

focus = \((h, k+p) = (0, 0+p) = (0, -2) \Rightarrow p = -2\)

Thus the equation is \(x^2 = 4(-2)y \Rightarrow x^2 = -8y\)

**Example 3:** Find the standard equation pf the parabola with focus at (1, 2) and directrix \(x = 3\).

**Solution:** The axis is perpendicular to the directrix line \(x = 3\), thus the axis is horizontal and goes through focus which is \(y = 2\).

The parabola has the form \((y - k)^2 = 4p(x - h)\)

The vertex is \((h, k)\)

The focus is \((h+p, k) = (1, 2) \Rightarrow h + p = 1, \text{ and } k = 2\)

The directrix is \(x = h - p = 3\)

We can solve \(h\) and \(p\) by using the system of linear equation \(h + p = 1\) and \(h - p = 3\)

Thus we have \(h = 2, k = 2, \text{ and } p = -1\)

The standard equation of the parabola with focus at \((1, 2)\) and directrix \(x = 3\) is \((y - 2)^2 = -4(x - 2)\)

**Example 4:** Write in **standard form** and find the **vertex** of the quadratic equation, and sketch the parabola

\[ y = x^2 - 2x - 3 \]

**Solution:** First, we need to rewrite this quadratic equation by

Completing the square: \( y = x^2 - 2x + 1 - 4 = (x - 1)^2 - 4 \)

Thus, \( y + 4 = (x - 1)^2 \) and that is the standard form of the parabola.

The vertex is \((-1, -4)\)